

BIBINUR DUISENGALIYEVA

**AUTOMORPHISMS OF THE DIFFERENTIAL POLYNOMIAL
ALGEBRAS AND FREE NOVIKOV ALGEBRAS**

ABSTRACT

of the dissertation of Duisengaliyeva B. for the degree of doctor of Philosophy
(PhD) in the specialty 6D060100 – Mathematics

Actuality of the dissertation theme. When studying all kinds of special mathematical structures of algebraic, geometric, or even non-mathematical origin, one can come across all kinds of symmetries that play an important role in the study of these structures. All these symmetries are taken into account and described in the language of automorphisms. For this reason, the group of automorphisms of a mathematical structure largely determines the situation in the structure itself. It is also well known that the group of automorphisms are used to classify and organize various mathematical objects. Therefore, the study of the group of automorphisms of algebraic systems and geometric varieties is one of the important and promising problems of modern mathematics. Free algebras have a fairly rich group of tame automorphisms. The question of the existence of wild automorphisms is very difficult.

In 1942, H.W.E. Jung proved that automorphisms of the polynomial algebra $k[x, y]$ in two variables over a field k of characteristic zero are tame. In 1953, W. van der Kulk generalized this result for the case of an arbitrary characteristic. A.J. Czerniakiewicz and L. Makar-Limanov proved that automorphisms of free associative algebra $k\langle x, y \rangle$ in two variables are tame. Moreover, the groups of automorphisms of algebras $k[x, y]$ and $k\langle x, y \rangle$ are isomorphic, that is,

$$\text{Aut}_k k[x, y] \cong \text{Aut}_k k\langle x, y \rangle.$$

D. Kozybaev, L. Makar-Limanov, U. Umirbaev proved that automorphisms of two-generated free right-symmetric algebras over arbitrary fields are also tame. A similar result for free Poisson algebras over a field of characteristic zero was obtained by L. Makar-Limanov, U. Turusbekova and U. Umirbaev, and it was also established that the group of automorphisms of this algebra is isomorphic to the group of automorphisms of the polynomial algebra $k[x, y]$. P. Cohn proved that automorphisms of free Lie algebras of finite rank are tame; an analogue of this result for free algebras of any homogeneous Schreier variety of algebras was obtained by J. Levin. Recall that Schreier varieties are varieties of all non-associative algebras, commutative and anticommutative algebras, Lie algebras and Lie superalgebras.

I.P. Shestakov and U.U. Umirbaev proved that groups of automorphisms of the polynomial algebra in three variables over a field of characteristic zero cannot be generated by all elementary automorphisms, that is, there are wild automorphisms. And also U.U. Umirbaev showed that the Anik automorphism

$$\delta = (x + z(xz - zy), y + (xz - zy)z, z)$$

of the free associative algebra $k\langle x, y, z \rangle$ over a field of characteristic zero is wild.

W. van der Kulk and M. Nagata proved that the group $Aut_k k[x, y]$ is represented as an amalgamated free product

$$Aut_k k[x, y] = A *_C B,$$

where A is the subgroup of affine automorphisms, B is the subgroup of triangular automorphisms and $C = A \cap B$. Similar results hold for free associative algebras, free Poisson algebras, free right-symmetric algebras over a field of characteristic zero.

Currently many mathematicians of Bulgaria, Russia, France, Germany, the United States and other countries, actively investigate automorphisms of free algebras and groups.

The object of the research is automorphisms and derivations of differential polynomial algebras and free Novikov algebras.

The aim of the research is to study automorphisms and derivations of differential polynomial algebras and free Novikov algebras.

Research methods. In the work uses interrelated methods for studying automorphisms and derivations of free algebras, applied of polynomial algebras, free associative algebras and free Poisson algebras. And also methods of the structural and combinatorial theory of differential polynomial algebras and free Novikov algebras are used.

Main results. The main results of the dissertation research are as follows:

- it was proved that the group of tame automorphisms of the differential polynomial algebras of rank two can be represented as an amalgamated free product of groups of affine automorphisms and triangular automorphisms;
- it is proved that any non-affine tame automorphism of the differential polynomial algebras of rank two is elementarily cancellable;
- an example of a wild automorphism (an analogue of the Anik automorphism) of partial differential polynomial algebras in two variables over a field of characteristic zero is constructed;
- an example of non-triangulable locally nilpotent derivation of partial differential polynomial algebras in two variables over a field of characteristic zero is constructed;
- the basic elements of free Novikov algebras in differential algebras are described;

- it was proved that the group of tame automorphisms of a free Novikov algebra of rank two can be represented as an amalgamated free product of groups of linear automorphisms and triangular automorphisms;

- an example of a wild automorphism (an analog of the Nagata automorphism) of free Novikov algebras in three variables over a field of characteristic zero is constructed;

- an example of non-triangulable locally nilpotent derivation of free Novikov algebras in three variables over a field of characteristic zero is constructed;

- it is proved that the differential dependence of a finite system of elements of differential polynomial algebras over a constructive field of characteristic zero is algorithmically recognizable;

- it is proved that the Novikov dependence of a finite system of elements of a free Novikov algebras over a constructive field of characteristic zero is algorithmically recognizable.

Scientific novelty. All the main results of the dissertation are new.

Theoretical and practical significance. The work is theoretical. The methods used and the results obtained can be applied to further study of differential polynomial algebras and free Novikov algebras. In addition, the results can be used when reading special courses on the theory of free algebras and their automorphisms.

Approbation of the results. The main results of the dissertation were reported:

- at the algebraic seminar of the Department of Algebra and Geometry of the Faculty of Mechanics and Mathematics of the L.N. Gumilyov Eurasian National University (Nur-Sultan, 2016-2020);

- at the seminar of the Faculty of Mechanics and Mathematics of the L.N. Gumilyov Eurasian National University (Nur-Sultan, 2018-2020);

- at the algebraic seminar of the Mathematical Faculty of Wayne State University (Detroit, USA, 2016);

- at the VI Congress of the Turkic World Mathematical Society (Nur-Sultan, 2017);

- at the international conference on algebra and mathematical logic "Maltsev Readings" (Novosibirsk, 2017);

- at the international algebraic conference dedicated to the 110th anniversary of the birth of Professor A.G. Kurosh (Moscow, 2018).

Publications. The main results of the dissertation were published in the form of articles in domestic and foreign journals, as well as in conference proceedings.

Structure and scope of work. The dissertation consists of an introduction, four sections, a conclusion and a list of references. The total volume of the dissertation is 67 pages. The list of references used, given at the end of the work, contains 62 items.

A summary of the work.

In the first section, we present information necessary for further work from the theory of differential polynomial algebras and the theory of free Novikov algebras.

The second section is devoted to the study of automorphisms and derivations of differential polynomial algebras over a field of characteristic zero. In subsection 2.1 introduces the concepts of highest homogeneous parts of elements and investigates some of their properties. In subsection 2.2 describes the group of tame automorphisms of the differential polynomial algebras of rank two.

Theorem 2.1 *The group of tame automorphisms of the algebra $A = k\{x, y\}$ is a free product of subgroups of affine automorphisms $Af_2(A)$ and triangular automorphisms $Tr_2(A)$ with amalgamated subgroup $C = Af_2(A) \cap Tr_2(A)$, that is,*

$$T(A) = Af_2(A) *_C Tr_2(A).$$

The main result of subsection 2.3 is the following theorem:

Theorem 2.2 *Any non-affine tame automorphism of the algebra $A = k\{x, y\}$ is elementarily cancellable.*

In subsection 2.4, we construct an example of a wild automorphism (an analogue of the Anik automorphism) of differential polynomial algebras in two variables with commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ in the case $m \geq 2$ over a field of characteristic zero.

Lemma 2.8 *Let $|\Delta| \geq 2$. The endomorphism δ of the algebra $A = k\{x, y\}$ defined as*

$$\delta(x) = x + w^{\delta_2}, \quad \delta(y) = y + w^{\delta_1},$$

where $w = x^{\delta_1} - y^{\delta_2}$, is an automorphism.

Theorem 2.3 *The automorphism δ of the algebra $A = k\{x, y\}$ is wild.*

In Section 2.5, we construct an example of a nontriangulable derivation of the algebra of differential polynomials in two variables with commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ in the case $m \geq 2$ over a field of characteristic zero.

Theorem 2.4 *Let $A = k\{x, y\}$ be the differential polynomial algebra in two variables x, y with a set of commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ over a field k of characteristic zero, and let $m \geq 2$. Then derivation*

$$D_1 = w^{\delta_2} \partial_x + w^{\delta_1} \partial_y,$$

where $w = x^{\delta_1} - y^{\delta_2}$, of the algebra $A = k\{x, y\}$ is not triangulable.

The third section is devoted to the study of automorphisms and derivations of free Novikov algebras over a field of characteristic zero. In subsection 3.1 describes the basic elements of free Novikov algebras in differential algebras.

Let u be a monomial of the differential polynomial algebra $k\{x_1, \dots, x_n\}$ in variables x_1, \dots, x_n with commuting derivations $\delta_1, \dots, \delta_m$ over a field k of characteristic zero. Let also $\deg(u)$ be the standard function of the degree of a monomial u with respect to variables x_1, \dots, x_n , and $d(u)$ be the differential degree of a monomial u with respect to derivations $\delta_1, \dots, \delta_m$.

Proposition 3.1 *The set of all differential monomials $u \in M$ with the condition $\deg(u) - d(u) = 1$ is a basis of the free Novikov algebra $N\langle x_1, \dots, x_n \rangle$.*

In subsection 3.2 describes the group of tame automorphisms of a free Novikov algebra of rank two.

Theorem 3.1 *Let $A = N\langle x, y \rangle$ be the free Novikov algebra in two variables x, y over a field k of characteristic zero. The group of tame automorphisms of the algebra A is a free product of subgroups of linear automorphisms $L_2(A)$ and triangular automorphisms $Tr_2(A)$ with amalgamation $C = L_2(A) \cap Tr_2(A)$, that is,*

$$T(A) = L_2(A) *_C Tr_2(A).$$

In subsection 3.3, we construct an example of a non-triangulable derivation of a free Novikov algebra of rank three.

Theorem 3.2 *Derivation*

$$D_1 = (2y \circ w_0) \partial_x + (z \circ w_0) \partial_y,$$

where $w_0 = \frac{1}{2}(2y \circ y - x \circ z - z \circ x)$, of the free Novikov algebra $N\langle x, y, z \rangle$ in three variables x, y, z over a field k of characteristic zero is not triangulable.

In subsection 3.4, we construct an example of a wild automorphism (an analogue of the Nagata automorphism) of a free Novikov algebra of rank three.

Theorem 3.3 *Automorphism*

$$\psi = \exp D_1 = (x + 2y \circ w_0 + (z \circ w_0) \circ w_0, y + z \circ w_0, z),$$

where $w_0 = \frac{1}{2}(2y \circ y - x \circ z - z \circ x)$, of the free Novikov algebra $N\langle x, y, z \rangle$ in three variables x, y, z over a field k of characteristic zero is wild.

The fourth section is devoted to the study of the differential algebraic dependence of a finite system of elements of the differential polynomial algebras and Novikov dependence of a finite system of elements of the free Novikov algebra. In subsection 4.1 defines an analogue of the Fox derivatives for differential polynomial algebras. Subsection 4.2 investigates the differential algebraic dependence of a finite system of elements of the differential polynomial algebras.

Theorem 4.1 Elements f_1, \dots, f_m of the field $B = k\langle x_1, x_2, \dots, x_n \rangle$ are differentially algebraically dependent if and only if $d(f_1), \dots, d(f_m)$ are left dependent over $B[\Delta]$.

Theorem 4.2 Let k be a constructive differential field. Then the differential algebraic dependence of a finite system of elements of a free differential field of rational functions over k is algorithmically recognizable.

In subsection 4.3, using the representation of free Novikov algebras in terms of differential polynomials, investigated the Novikov dependence of a finite system of elements of a free Novikov algebra.

Theorem 4.3 Let f_1, f_2, \dots, f_p be elements of the free Novikov algebra $N\langle x_1, x_2, \dots, x_n \rangle$. The elements f_1, f_2, \dots, f_p are Novikov dependent if and only if they are differentially algebraically dependent in $k\{x_1, x_2, \dots, x_n\}$.

Corollary 4.2 Novikov dependence of a finite system of elements of a free Novikov algebra $N\langle x_1, x_2, \dots, x_n \rangle$ over a constructive field k of characteristic zero is algorithmically recognizable.

Thus, we solved the tasks to achieve the goal of the dissertation research.